Polynomials

The general form for a polynomial is

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x + a_0 .$$

Generally, when we work with polynomials, we are restricted to the real numbers. You are well aware that a quadratic polynomial can have two distinct real zeros, one double zero, or no real roots. Factoring or the quadratic formula can be used to find all zeros.

A cubic polynomial can have one real and two complex roots, or three real roots. Factoring, if one zero is fairly obvious, along with long or synthetic division, can be used to find roots. There is a cubic formula, but too complicated to learn and use.

A quartic polynomial can have four real roots (possible two distinct and one double, two double zeros, or one zero with multiplicity 4), or two real and two complex, or no real roots and 4 complex roots. Like the cubic, factoring works, but the quartic formula exists but is not useful.

Roots and Coefficients. If the coefficients of a polynomial are rational, there can be individual rational roots, but irrational roots will always occur in conjugate pairs. For example, suppose 1, -2, and $2-\sqrt{3}$ are zero of P(x) and we are told that the coefficients are rational, then $2+\sqrt{3}$ is also a zero. We know too that zeros and factors are related,

so $$\begin{aligned} P(x) &= k(x-1)(x+2)\left(x-\left(2-\sqrt{3}\right)\right)\left(x-\left(2+\sqrt{3}\right)\right) \\ &= k\left(x^2+x-2\right)\left(x^2-4x+1\right) = k\left(x^4-3x^2-5x^2+9x-2\right) \end{aligned}.$$

Ignoring the constant k since it does not affect the zeros and notice the following. The sum of the zeros is $1+(-2)+\left(2-\sqrt{3}\right)+\left(2-\sqrt{3}\right)=3$, that the sum of all possible products for zeros taken two at a time is

$$(1(-2))+\left(1\left(2-\sqrt{3}\right)\right)+\left(1\left(2+\sqrt{3}\right)\right)+\left(-2\left(2-\sqrt{3}\right)\right)+\left(-2\left(2+\sqrt{3}\right)\right)+\left(\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)\right)$$

$$-2 \qquad +2-\sqrt{3} \qquad +2+\sqrt{3} \qquad -4+2\sqrt{3} \qquad -4-2\sqrt{3} \qquad +4-3 \quad =-5$$

and the sum of the products of the roots taken three at a time is

$$\left((1)(-2)\left(2-\sqrt{3}\right)\right)+\left((1)(-2)\left(2+\sqrt{3}\right)\right)+\left((1)\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)\right)+\left((-2)\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)\right)=$$

$$-4+2\sqrt{3} \qquad -4-2\sqrt{3} \qquad +4-3 \qquad -8+6 \qquad =-9$$

and finally the product of all 4 zeros is $(1)(-2)\left(2-\sqrt{3}\right)\left(2+\sqrt{3}\right)=-2$. The polynomial now is $P(x)=x^4-Ax^3+Bx^2-Cx+D$, where A is the sum of the zeros, B the sum of the products taken two at a time, C the sum of the products taken 3 at a time, and D the sum of the products taken 4 at a time. This pattern continues for higher degree polynomials.

You will see it most often in problems where you are told something about the polynomial and you will use or find the sum and or product of the zeros. Not that you do not have to find the roots to know their sum or product. A simple example might be as follows: Given $P(x) = x^3 - 5x^2 + 2x + 12$, find the sum (or product) of the zeros. The slow way is to actually find the zeros by trial and error maybe (3, $1+\sqrt{5}, 1-\sqrt{5}$) and then the sum (or product). The fast answer is just 5 for the sum and -12 for the product.

Rational Roots Theorem. If the coefficients of a polynomial are integers, then the rational roots, if any, will have to be of the form ${}^{\pm p}\!/_{\pm q}$, where *p* is a integer factor of the constant term and *q* is an integer factor of the leading term. Let's use a simple example to see why this works. Suppose

$P(x) = (ax-b)(cx-d)(gx-h) = acgx^3 - (ach+adg+bcg)x^2 + (adh+bch+bdg)x - (bdh)$

The rational zeros of the polynomial were $\frac{b}{a}, \frac{d}{c},$ and $\frac{h}{g}$. Notice that the numerators are all factors of the constant term *(bdh)* and the numerators are all factors of the leading coefficient (*acg).*

Synthetic Division and Remainder Theorem

Synthetic division is based on long division and is useful in finding the value of a polynomial for a given x as well as the factors of the polynomial. (If the value of a polynomial for a given number *a*, then *x – a* is a factor.

Long Division

$$
\begin{array}{r}
x^2 + 3x + 7 \\
(x-2)\overline{)x^3 + x^2 + x - 14} \\
\underline{x^3 - 2x^2} \\
+3x^2 + x - 14 \\
\underline{3x^2 - 6x} \\
7x - 14 \\
\underline{7x - 14} \\
0
\end{array}
$$

Synthetic Division

In Synthetic Division, write the number being checked (as a possible zero perhaps) and then the coefficients as follows:

$\underline{2}]\quad 1 \quad 1 \quad 1 \quad -14$

Now leave a row blank and draw a horizontal line below the coefficients.

$$
\begin{array}{r|rrrr}
2 & 1 & 1 & 1 & -14 \\
 & & & & \\
\hline
\end{array}
$$

Bring down the leading coefficient and then multiply it by the zero candidate, placing this product in the space below the next coefficient.

$$
\begin{array}{r|rrrr}
2 & 1 & 1 & 1 & -14 \\
 & & 2 & & \\
\hline
 & 1 & & &
\end{array}
$$

Now add, write the total below, and repeat.

2]	1	1	1	–14
		2	6	14
	1	3	7	0

Notice that the final sum is zero, telling us that 2 is a zero. But the nice thing about synthetic division, is that long division, the other coefficients are the coefficients of the quotient polynomial.

The Remainder Theorem states that when dividing a polynomial by a factor of the form $(x-a)$ there is a quotient polynomial and a constant remainder. The remainder is P(a) since $P(x)=(x-a)\cdot Q(x)+r \Rightarrow P(a)=(a-a)Q(a)+r=r$.

Descartes' Rule of Sign Changes. In a polynomial with real coefficients, the number of positive real roots equals the number of sign changes in the coefficients, or that number less multiples of 2. The number of negative real roots is the number of sign changes (less multiples of 2) in the polynomial P(-x). For example: $P(x)=x^5-9x^4-24x^2+5x+126$ has only two sign changes, so there are either 2 or 0 positive real roots. $P(-x)=-x^5+9x^3-24x^2-5x+126$ has 3 sign changes so there are either 3 or 1 negative real roots. Turns out that there are two positive real roots, one rational and one irrational and only one negative (irrational) root. The other two roots are complex.

Miscellaneous Facts;
It is obvious in any polynomial that P(0) is the y-intercept and equally as obvious that P(1) is the sum of all the coefficients.

Problems:

1. Given the cubic polynomial $P(x)=x^3-7x^2-4x+28$. Two of the zeros are additive inverses. Find the zeros.
2. If $p(x)=ax^2+bx+c$ leaves a remainder of 4 when divided by x, a remainder of 3 when divided by x + 1, and a remainder of 1 when divided by x – 1, then p(2) is?
3. If P(x) is a polynomial with rational coefficients and roots at 0, 1, $\sqrt{2}$, and $1-\sqrt{3}$, then the degree of p(x) is at least?
4. When Madison's dog chewed up her mathematics assignment, one particular equation was ripped apart. I found a piece of the beginning of the equation and a piece at the end, but the middle was missing. The beginning piece was x^5-9x^4+ and the ending piece was $+11=0$. Fortunately the teacher had promised that all of the roots would be integers. How many times is -1 a root? [Furman ????]
5. The following is a polynomial. Find the sum of the squares of its coefficients. $\sqrt[3]{x^9-3x^8+18x^7-28x^6+84x^5-42x^4+98x^3+72x^2+15x+1}$. **FURMAN**
6. If a cubic polynomial p(x) has roots at -1, 2, and 3, and if p(0) = 1, then the remainder when p(x) is divided by x – 1 is:
7. If 2 is a solution of $x^3+hx+10=0$, then h equals:

8. The number of distinct real solutions of the equation $4x^3 - 8x^2 + 5x - 1 = 0$ is:

9. What is the sum of the squares of the roots of $x^4 - 5x^2 + 6 = 0$

10. For how many integers N is $N^4 + 6N < 6N^3 + N^2$?

11. How any times does the graph of $f(x) = x^3 - x^2 + 2x + 4$ cross the x axis?

12. Madison's dog chewed on her homework before she could finish it. The fragment saved from the horrible canine's mouth reveal only the two terms of highest degree of the polynomial p(x). It looked like $p(x) = x^{18} - 3x^{17} + \cdots$. Madison found roots 1,2,3,…17. What was the missing root? **FURMAN 2000 SR#27**

13. Find the largest x so that $|x| < \frac{1}{2}$ and $x^4 + x^3 - 3x^2 + x = 0$. **FURMAN 2000 SR #8**

14. The sum of the solutions of $(x+2)^3 + (3x-6)^3 = (4x-4)^3$ is *k*, Find *k*. **ARML 1980 Relay 2-3**.

15. The real value of *x* which satisfies $x^3 + (x-1)^3 + (x-2)^3 + (x-3)^3 + (x-4)^3 + (x-5)^3 = 3^3$ is *k*. Find *k*. **ARML 1980 Relay 2-4.**

16. Find the four values of *x* for which $(x-3)^4 + (x-5)^4 = -8$. **ARML 1978 Team 8.**

17. The equations $(x-2)^4 - (x-2) = 0$ and $x^2 - kx + k = 0$ have two roots in common. Find the value of *k*. **ARML 1978 Ind 5**

18. Find the remainder that results when $(x+1)^5 + (x+2)^4 + (x+3)^3 + (x+4)^2 + (x+5)$ is divided by $x+2$. **ARML 1977 Team 2**

19. If $a, 3a, 5a, b, b+3, b+5$ are all roots of a 4th degree polynomial equation, where $0 < a < b$, compute all possible values of *a*.

20. Let $P(x)$ be a polynomial whose degree is 1996. If $P(n) = \frac{1}{n}$ for $n = 1, 2, 3, \cdots, 1997$, compute the value of $P(1998)$. **ARML 1997 Team 9**

21. Let *r* and *s* denote the two real roots of $x^2 - \sqrt{5}x + 1 = 0$. Then $r^8 + s^8$ is an integer. Determine this integer. **ARML 1981 Team 4**

22. There are two values for *k* for which the cubic polynomial $2x^3 - 9x^2 + 12x - k$ has a double root. What are these two values? **ARML 1981 Team 5**

23. Determine the sum of the y-coordinates of the four points of intersection of $y = x^4 - 5x^2 - x + 4$ and $y = x^2 - 3x$. **ARML 2006 Ind 6.**

24. Determine the sum of the y-coordinates of the three points of intersection of $y = x^2 - x - 5$ and $y = \frac{1}{x}$.