Perfect Squares:

There are several strategies for solving problems that involve perfect squares.

Differences of two squares.

On the 2000 State Math Contest question 3 asked:
Let *p* be an odd prime number. How many ordered pairs of positive integers (x, y) with $x < y$ satisfy the condition $y^2 - x^2 = p$?

Since the difference of two perfect squares always factors, this is what we will do. $(y-x)(y+x) = p$. But since *p* is prime, its only factors are 1 and *p*, so $y - x = 1$ and $y + x = p$. Solving this system we find that $2y = p+1$ which works for all odd primes, and $2x = p-1$, which works for all odd primes as well. Thus there are infinitely many solutions. $p = 3 \Rightarrow y = 2, x = 1;$ $p = 5 \Rightarrow y = 3, x = 2;$ etc.

Completing the Square

For how many positive integers *n* is $n^2 + 18n$ a perfect square? Well

$$n^2 + 18n = k^2 \Leftrightarrow n^2 + 18n + 81 = k^2 + 81$$

$$\Leftrightarrow (n+9)^2 - k^2 = 81 \Rightarrow ((n+9)+k)((n+9)-k) = 81.$$

Look at the factors of 81 in pairs: (1,81), (3,27), and (9,9). Let $(n+9) - k = 1$ and $(n+9) + k = 81$ and subtract to find *k*. In this first case, $k = 40$. The other pairs yield $k = 12$ and 0, making $n = 32, 6, 0$, but we cannot use 0. Checking we see that $32^2 + 18 \cdot 32 = 1600 = 40^2$ $6^2 + 18 \cdot 6 = 144 = 12^2$.

Writing as a square:

In this method take the desired quantity and write it as the square of some number, perhaps involving some or all of the original expression. Then see what makes sense.

In this problem from the 2002 NC State Math Contest, we are asked to find the four values for which $n^2 + n + 109$ is a perfect square.

$$n^2 + n + 109 = (n+k)^2 = n^2 + 2nk + k^2$$

$$\Rightarrow n + 109 = 2nk + k^2 \Rightarrow n(1-2k) = k^2 - 109$$ Now check values of *k* from 1 through 10.

$$\Rightarrow n = \frac{109 - k^2}{2k - 1}$$

k	1	2	3	4	5	6	7	8	9	10
$n=\dfrac{109-k^2}{2k-1}$	108	35	20	13.2..	9.3..	6.6..	4.6..	3	1.6..	0.47..

The integer quantities for n, paired with the integer values for k yield 109, 37, 23, and 11.

Problems:

1. The number of solutions of the equation $x^2 - y^2 = 63$ in positive integers is:
FURMAN 2003 Jr #29

2. How many integers between 500 and 1500 are perfect squares?
FURMAN 2001 JR #1

3. Find the sum of all positive integers n so that $2001 + n^2$ will be a perfect square.
NC SMC 2001 INT8.

4. Find the sum of all positive integers n so that $2004 + n^2$ will be a perfect square.

5. How many perfect squares are divisors of the product $1!\cdot 2!\cdot 3!\cdot 4!\cdot 5!\cdot 6!\cdot 7!\cdot 8!\cdot 9!$?
AMC12 2003 #23

6. For what positive integer values of n is $n^2 - 19n + 99$ a perfect square?
AIME 1999

7. For how many positive integers n is $n^2 - 2004n$ a perfect square?