Partial Fractions

Since elementary school you have known how to add fractions and since Algebra you have been able to add fractions with variables. The process of taking a single fraction and breaking it up into two or more terms is less obvious and often is not learned by students until they need this skill in a calculus class to do integration problems. It is, however, a useful skill.

Suppose we have the fraction $\frac{2}{x(x-1)}$. This is obviously equal to $\frac{2}{x}\cdot\frac{1}{x-1}$, or $\frac{1}{\mathrm{x}}\cdot\frac{2}{x-1}$, or any other product where the product of the numerators is 2. This decomposition is obviously not unique. On the other hand, if we try to express the original fraction as a sum (or difference) of two fractions, this decomposition is unique.

Suppose $\frac{2}{x(x-1)}=\frac{A}{x}+\frac{B}{x-1}$, where *A* and *B* are real numbers. Then $\frac{A}{x}+\frac{B}{x-1}=\frac{A(x-1)}{x(x-1)}+\frac{Bx}{x(x-1)}=\frac{(A+B)x-A}{x(x-1)}$. Now for this last expression to equal the original fraction, the numerators must be equal, so $2=(A+B)x-A$, for all values of *x*. This means that the coefficient of *x*, which is *A* + *B*, must equal 0 and –*A* must = 2. Therefore $A=-2, B=2$, so $\frac{2}{x(x-1)}=\frac{-2}{x}+\frac{2}{x-1}$. In calculus, we use this decomposition to evaluate the integral $\int\frac{2}{x(x-1)}dx=\int\frac{2}{x-1}dx-\int\frac{2}{x}dx=2\cdot\ln(x-1)-2\cdot\ln(x)+C=2\cdot\ln\left(\frac{x-1}{x}\right)+C$.

In a problem like the following, this method can be used to greatly simplify the solution.

Problem: Find the sum of the first 200 terms of the series $\frac{1}{1\cdot2}+\frac{1}{2\cdot3}+\frac{1}{3\cdot4}+\cdots+\frac{1}{n\cdot n+1}+\cdots$.

Solution: Rewrite each term, using partial fractions as follows:
$\frac{1}{n(n+1)}=\frac{A}{n}+\frac{B}{n+1}\Rightarrow A(n+1)+B(n)=1\Rightarrow A=1, B=-1$, so the series can be rewritten as $\left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$. Now notice that the terms, upon re-association, is $\frac{1}{1}+\left(\frac{-1}{2}+\frac{1}{2}\right)+\left(\frac{-1}{3}+\frac{1}{3}\right)+\cdots\left(\frac{-1}{n}+\frac{1}{n}\right)-\frac{1}{n+1}=1-\frac{1}{n+1}$. This makes the answer to the problem simply $1-\frac{1}{201}=\frac{200}{201}$.

As long as the factors in the denominator are linear, the numerators will all be constants. In general, the numerator will always be in degree one less than the degree of the numerator.

Suppose we want to write the fraction $\frac{2n+3}{n(n^2+1)}$ as the sum of two other fractions. To do this write this fraction as $\frac{A}{n}+\frac{Bn+C}{n^2+1}$ and then add these two to get $\frac{A(n^2+1)+(Bn+C)n}{n\left(n^2+1\right)}=\frac{\left(A+B\right)n^2+Cn+A}{n\left(n^2+1\right)}$. The numerator of this fraction must equal $2n+3$, so $A+B=0, C=2, A=3$, so B must be -3. So $\frac{2n+3}{n(n^2+1)}=\frac{3}{n}+\frac{-3n+2}{n^2+1}$.

Problems:

1. Find the sum of the first 100 terms of the series $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots\frac{1}{n\cdot(n+1)}$.

2. Find the sum of the first 50 terms of the series $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots\frac{1}{(2n-1)\cdot(2n+1)}$.

3. Find the sum of the series $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\frac{1}{3\cdot 4\cdot 5}+\cdots+\frac{1}{98\cdot 99\cdot 100}$.

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4. How many terms of the series $\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\frac{1}{5\cdot 6}+\cdots\frac{1}{\left(n+1\right)\left(n+2\right)}$ must be added to be greater than 0.4999?

5. Find the sum of the first 50 terms of the series $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots+\frac{1}{\left(n^2-1\right)}+\cdots, n\geq 2.$ What is the sum of the infinite series?

6. Find the sum of the first 50 terms of the series $\frac{1}{6}+\frac{1}{11}+\frac{1}{20}+\cdots+\frac{1}{n^2+n}+\cdots, n\geq 2.$ What is the sum of the infinite series?

7. The series $\sum_{n=2}^{\infty}\frac{1}{n^2}$ is sandwiched in between the two series in 5 and 6. Can you find the sum of this infinite series by the methods of this section? Why or why not?