Finite Differences and Recurrence Relations

Finding a closed formula for a set of ordered pairs is a useful skill. There are two methods that work nicely. The first is finite differences and the second recurrence relations.

In order to fit a function to a set of points, it is helpful to know the type of function you are dealing with. This is where finite difference can help. If the *x*-values are sequential and only one unit apart, the finite differences are simply the differences in the dependent values of a function. Let's look at the following set of linear data first.

x	y	
1	3	
		4
2	7	
		4
3	11	
		4
4	15	

It should be fairly obvious that the first finite differences are constant, and that this is the slope of the linear function that would include these points. These finite differences are simply the change in y while the change in x each time is just 1. We know the function then is $f(x) = 4x + b$ and it is each to substitute in any value to determine that b is -1.

Now let's look at some non-linear data.

1	5		
		0	
2	5		2
		2	
3	7		2
		4	
4	11		2
		6	
5	17		

The first finite differences are not constant, so this is clearly not a linear function. The second differences are constant, and this tells us that the function is quadratic. To see why, let's look at a generic quadratic function of the form $y = ax^2 + bx + c$.

Three points with successive x-coordinates might be

$$\left(n-1,\ f(n-1)\right),\ \ \left(n,\ f(n)\right),\ \left(n+1,\ f(n+1)\right).$$

If we put these in a table, we get

Written & Compiled by John Goebel, NCSSM Problem Solving Course, 2006

$n-1$	$a(n-1)^2+b(n-1)+c$	$=an^2+(b-2a)n+(a-b+c)$		
			$2an+(b-a)$	
n	an^2+bn+c	$=an^2+bn+c$		$2a$
			$2an+(a+b)$	
$n+1$	$a(n+1)^2+b(n+1)+c$	$=an^2+(b+2a)n+(a+b+c)$		

The first differences were not equal (as long as a is not zero), but the second differences are not only constant, but equal to 2a.

Returning to our specific example, we now know that the function is quadratic and a = 1. Simple substitution can be used to determine the values of b and c as -3 and 7.

If the function is cubic, the third difference will be constant and will be 6a. Look at the following example:

X	Y	△1	△2	△3
1	-3			
		4		
2	1		8	
		12		6
3	13		14	
		26		6
4	39		20	
		46		6
5	85		26	
		72		
6	157			

The 3rd differences are constant and equal to 6. This is actually 3!a or 6a, so a = 1. The second differences are 6ax + 2b, so when x = 2, (6)(1)(2) + 2b = 8, making b = -2. Simple substitution with two points can be used to find the last two coefficients.

This function is $f(x)=x^3-2x^2+3x-5$,

Exponential functions are a little more difficult. The general form for an exponential function is $f(x)=a\cdot b^x+c$. Let's look at the following example.

x	y	△1	△2
1	10		
		12 = 12(1)	
2	22		24 = 24(1)
		36 = 12(3)	
3	58		72 = 24(3)
		$108 = 12(3^2)$	
4	166		$216 = 24(3^2)$
		$324 = 12(3^3)$	
5	490		$648 = 24(3^3)$
		$972 = 12(3^4)$	
6	1462		

The first and second differences are not constant, and never will be. Notice, however, that there is a common factor of 12 in the first difference and a common factor of 24 in the second differences. From this we can conclude that the base of our function is 3. The function now is of the form $y=a\cdot 3^x+c$. Again, simple substitution can be used to find that a is 2 and c is 4, so the exponential function that fits these points is $f(x)=2\cdot 3^x+4$.

Recurrence Relations

Often we are given a recursive formula to generate values of a sequence and we would like to find the closed form. You are familiar with arithmetic and geometric sequences and know how to find both the recursive and closed formulas for the terms of such sequences. If, however, the recurrence relation is a little more complicated, it is not as easy to find the closed formula.

Take for example the simple case where you both multiply and add a number at each step in the sequence. So $t_0 = a$ and at each step we will multiply by *r* and then add *d*, making the recursive equation $t_{n+1} = t_n \cdot r + d$. This is called a first order recurrence relation, and it is easy to solve algebraically. List the first several terms and you will see the closed formula:

Term	Value
0	A
1	$Ar + d$
2	$(Ar + d)r + d = Ar^2 + dr + d$
3	$(Ar^2 + dr + d)r + d = Ar^3 + dr^2 + dr + d$
4	$(Ar^3 + dr^2 + dr + d)r + d = Ar^4 + \sum_{j=0}^{3} dr^j$

So if we apply the closed formula for the ending geometric series, we get the final closed formula for the entire sequence:

$$t_n = A \cdot r^n + d\frac{r^n - 1}{r - 1}$$

Recurrence equations of the form $t_n = A \cdot t_{n-1} + B \cdot t_{n-2}$ can be rewritten in the form

$$t_n - A \cdot t_{n-1} - B \cdot t_{n-2} = 0,$$

and such equations are called second order homogeneous recurrence equations or relations. Clearly you must know two initial values and the values of A and B in order to recursively generate the terms of the sequence. Finding the closed form for the terms is another matter entirely.

First, we will assume that there are solutions of the form $t_n = c \cdot r^n$, so that our equation becomes

$$cr^n - Acr^{n-1} - Bcr^{n-2} = 0$$

which is equivalent to

$$cr^{n-2}\left(r^2 - Ar - B\right) = 0$$

so we can solve the second factor of this last equation to find values for *r*. The two solutions for this equation are

$$\frac{A \pm \sqrt{A^2 + 4B}}{2}.$$

The closed form then for $t_n = c_1 \cdot r_1^{\ n} + c_2 \cdot r_2^{\ n}$, where
$r_1 = \frac{A + \sqrt{A^2 + 4B}}{2}, r_2 = \frac{A - \sqrt{A^2 + 4B}}{2}$.

One very familiar recurrence relation is the Fibonacci sequence. Here $t_0 = 0, t_1 = 1,$ and $t_n = t_{n-1} + t_{n-2}$. The first few terms are 0, 1, 1, 2, 3, 5, 8, 13, 21, … If we use the solution outlined above, you get $cr^n - cr^{n-1} - cr^{n-2} = 0$, so that $cr^{n-2}\left(r^2 - r - 1\right) = 0$. The solutions to the second factor are $r_1 = \frac{1 + \sqrt{1+4}}{2} = \frac{1+\sqrt{5}}{2}, r_2 = \frac{1 - \sqrt{1+4}}{2} = \frac{1-\sqrt{5}}{2}$. Now we have $t_n = c_1 \cdot r_1^{\ n} + c_2 \cdot r_2^{\ n} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$. Now take the first two given values to find the values for c_1, c_2. We get the system of equations:

$$\begin{aligned} 0 &= c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0 = c_1 + c_2 \Rightarrow c_2 = -c_1 \\ 1 &= c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 - c_1\left(\frac{1-\sqrt{5}}{2}\right)^1 = c_1\left(\sqrt{5}\right) \Rightarrow c_1 = \frac{1}{\sqrt{5}}, c_2 = -\frac{1}{\sqrt{5}} \end{aligned}.$$

This means then that $t_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

Austin's Problem. The first term of a sequence is 0 and the second term 1. Each successive term is defined as follows: $t_n = 4 \cdot t_{n-1} - t_{n-2}, n \geq 2$. Prove that for and $n \geq 1, t_{n-1} \cdot t_{n+1} = \left(t_n\right)^2 - 1$.

Solution. First, notice that the first few terms of this sequence are 0, 1, 4, 15, 56, 208, … and it looks like the result is certainly true as $0 \cdot 4 = 1^2 - 1 = 0$, $1 \cdot 15 = 4^2 - 1 = 15$,

$4 \cdot 56 = 15^2 - 1 = 224$, …. Certainly one way to solve this is to find a closed form for the terms of this sequence. As in the Fibonacci sequence, this is a homogeneous recurrence relation with $cr^n - 4cr^{n-1} + cr^{n-2} = 0$, so $cr^{n-2}\left(r^2 - 4r + 1\right) = 0$. The solutions to the second factor are

$r_1 = \frac{4+\sqrt{16-4}}{2} = \frac{4+2\sqrt{3}}{2} = 2+\sqrt{3} =, r_2 = \frac{4-\sqrt{16-4}}{2} = \frac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$. Now we have $t_n = c_1 \cdot r_1^{\,n} + c_2 \cdot r_2^{\,n} = c_1\left(2+\sqrt{3}\right)^n + c_2\left(2-\sqrt{3}\right)^n$. Now take the first two given values to find the values for c_1, c_2. We get the system of equations:

$$\begin{aligned}
0 &= c_1\left(2+\sqrt{3}\right)^0 + c_2\left(2-\sqrt{3}\right)^0 = c_1 + c_2 \Rightarrow c_2 = -c_1 \\
1 &= c_1\left(2+\sqrt{3}\right)^1 - c_1\left(2-\sqrt{3}\right)^1 = c_1\left(2\sqrt{3}\right) \Rightarrow c_1 = \frac{1}{2\sqrt{3}}, c_2 = -\frac{1}{2\sqrt{3}}
\end{aligned}.$$

This means then that $t_n = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^n - \left(2-\sqrt{3}\right)^n\right)$. Now to establish the desired result, that $n \ge 1, t_{n-1} \cdot t_{n+1} = \left(t_n\right)^2 - 1$.

First $t_{n-1} = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n-1} - \left(2-\sqrt{3}\right)^{n-1}\right)$ and $t_{n+1} = \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right)$. It really helps here to realize that $2-\sqrt{3} = \frac{1}{2+\sqrt{3}} = \left(2+\sqrt{3}\right)^{-1}$. So the product

$$\begin{aligned}
t_{n-1} \cdot t_{n+1} &= \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n-1} - \left(2-\sqrt{3}\right)^{n-1}\right) \cdot \frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2-\sqrt{3}\right)^{n+1}\right) \\
&= \frac{1}{12}\left(\left(2+\sqrt{3}\right)^{n-1} - \left(2+\sqrt{3}\right)^{1-n}\right)\left(\left(2+\sqrt{3}\right)^{n+1} - \left(2+\sqrt{3}\right)^{-(n+1)}\right) \\
&= \frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n} - \left(2+\sqrt{3}\right)^{-2} - \left(2+\sqrt{3}\right)^{2} + \left(2+\sqrt{3}\right)^{-2n}\right) \\
&= \frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n} - \left(2-\sqrt{3}\right)^{2} - \left(2+\sqrt{3}\right)^{2} + \left(2+\sqrt{3}\right)^{-2n}\right) \\
&= \frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n} - \left(7-4\sqrt{3}\right) - \left(7+4\sqrt{3}\right) + \left(2+\sqrt{3}\right)^{-2n}\right) \\
&= \frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n} - 14 + \left(2+\sqrt{3}\right)^{-2n}\right)
\end{aligned}$$

Now look at

$$\begin{aligned}(t_n)^2-1&=\left(\frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^n-\left(2-\sqrt{3}\right)^n\right)\right)^2-1\\&=\left(\frac{1}{2\sqrt{3}}\left(\left(2+\sqrt{3}\right)^n-\left(2+\sqrt{3}\right)^{-n}\right)\right)^2-1\\&=\frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n}-2+\left(2+\sqrt{3}\right)^{-2n}\right)-\frac{12}{12}\\&=\frac{1}{12}\left(\left(2+\sqrt{3}\right)^{2n}-14+\left(2+\sqrt{3}\right)^{-2n}\right)\end{aligned}$$

These two quantities are equal, so that proves the given result.

Problems.

1. Make an argument to show that the third differences for cubic data is actually 6a.

2. Find the sum of the squares of the first 100 positive integers.

3. Find the sum of the cubes of the first 50 positive integers.

4. Find the sum $1-4+9-16+\cdots-100^2$.

5. The Lucas sequence begins with 2, then 1, and then the remaining terms are found as in the Fibonacci sequence. Write the recursive and closed forms for the terms of this sequence.

6. The first term of a sequence is 0 and the second is 1. For all $n \ge 2, t_n = 3\cdot t_{n-1} - 2\cdot t_{n-2}$. Find the first 5 terms of the sequence, the recursive, and the closed form for the terms of this sequence.

7. Sequence $(a_1, a_2, a_3, ...)$ is defined recursively by $a_1 = 0, a_2 = 100$ and $a_n = 2a_{n-1} - a_{n-2} - 3$. Find the greatest term in the sequence $(a_1, a_2, a_3, ...)$.
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8. Find k given that $\begin{cases} f(0) = k \\ f(n) = f(n+1) - 3n - 2 \end{cases}$ and $f(-50) = 4000$.
   NC SMC 2003 Integer 11.