Distance Formulas

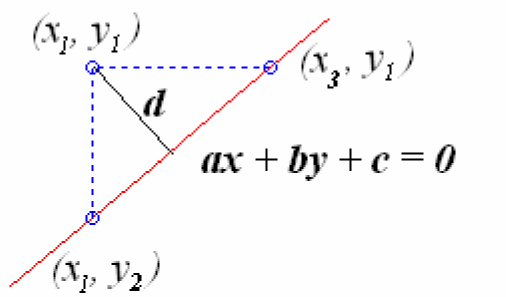
The Pythagorean Distance formula for the shortest distance between two points (x_1, y_1) and (x_2, y_2) is $d = \sqrt{(x_2 - x_2)^2 - (y_2 - y_1)^2}$.

The comparable formula for distance in space between the points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) is $d = \sqrt{(x_2 - x_2)^2 - (y_2 - y_1)^2 + (z_1 - z_2)^2}$.

The distance from a point (x_1, y_1) to a line $ax + by + c = 0$ is one of those problems that seems easy, but is very time consuming, unless you know the following formula.

$$d = \left| \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}} \right|$$

The following proof illustrates why it pays to know the formula.



Drop perpendiculars to the line from the given point (x_1, y_1). Now $y_2 = -\frac{a}{b}x_1 - \frac{c}{b}$ and $x_3 = -\frac{a}{b}x - \frac{c}{b}$. The triangle in the picture is a right triangle, so the area can be found two ways. The product of the legs will equal the altitude (d) times the hypotenuse.

$$\left| y_1 + \frac{a}{b}x_1 + \frac{c}{b} \right| \times \left| x_1 + \frac{b}{a}y_1 + \frac{c}{a} \right| = d \times \sqrt{\left(y_1 + \frac{a}{b}x_1 + \frac{c}{b} \right)^2 + \left(x_1 + \frac{b}{a}y_1 + \frac{c}{a} \right)^2}\text{, so}$$

$$d = \frac{\left| y_1 + \frac{a}{b}x_1 + \frac{c}{b} \right| \times \left| x_1 + \frac{b}{a}y_1 + \frac{c}{a} \right|}{\sqrt{\left(y_1 + \frac{a}{b}x_1 + \frac{c}{b} \right)^2 + \left(x_1 + \frac{b}{a}y_1 + \frac{c}{a} \right)^2}} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Example: Find the distance between the point (2,-3) and the line $3x-4y-12=0$. The distance is $d=\left|\frac{3(2)-4(-3)-12}{\sqrt{3^2+4^2}}\right|=\frac{|6|}{5}=\frac{6}{5}$.

The distance from a point in space to a plane can be found using an extension of this formula. The distance from the points $\left(x_1, y_1, z_1\right)$ to the plane $ax+by+cz+d=0$ is $\frac{\left|ax_1+by_1+cz_1+d\right|}{\sqrt{a^2+b^2+c^2}}$

Problems.

1. What is the closest the line $4x+5y=20$ comes to the origin?

2. What is the distance between the points $(2, f(2))$ and $\left(f(2), 2\right)$ if $f(x)=\sqrt{4x+1}$.

3. How far from the origin are the points of intersection of the conics $x^2+7y^2=47$ and $2x^2-4y^2=28$?

4. What is the shortest distance between the circle $x^2+y^2=25$ and the line $3x+4y=48$?

5. Find the distance between the parallel lines $2x-3y=12$ and $2x-3y=36$.

6. A triangle has vertices (2,3), (6,-2), and (-1,-4). Find the area of the triangle.

7. Find the shortest distance between the parallel planes $3x+4y-5x=12$ and $3x+4y-5z=20$.

8. What is the closest that the line $y=\frac{4}{7}x+\frac{1}{5}$ comes to a lattice point? **DUKE 2003 Team10.**

9. What is the closest the plane $z=\frac{1}{3}x+\frac{3}{5}y+\frac{4}{15}$ comes to a lattice point?

10. Under what conditions would the line $Ax+By+C=0$ be guaranteed to hit a lattice point?

11. The circles $x^2+y^2+10x-6y+18=0$ and $x^2+y^2-14x+4y+44=0$ have two internal tangents. On each the distance between the points of tangency are equal. Find this distance.

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