Complex Numbers

Solutions:

1. Show that the power rule works for n = 2, and n = 3.

$$\left[r\left(\cos\theta + i\sin\theta\right)\right]^2 = r^2 \cdot \left(\cos\theta + i\sin\theta\right)^2 =$$
$$r^2\left(\cos^2(\theta) + 2i\cos(\theta)\sin(\theta) + i^2\sin^2(\theta)\right) =$$
$$r^2\left(\cos^2(\theta) - \sin^2(\theta) + 2i\sin(\theta)\cos(\theta)\right) =$$
$$r^2\left(\cos(2\theta) + i\sin(2\theta)\right)$$

$$\left[r\left(\cos\theta + i\sin\theta\right)\right]^3 = r^2\left(\cos(2\theta) + i\sin(2\theta)\right)\left(r\left(\cos(\theta) + i\sin(\theta)\right)\right) =$$
$$r^3\left[\cos(2\theta)\cos(\theta) - \sin(\theta)\sin(2\theta) + i\left(\sin(\theta)\cos(2\theta) + \sin(2\theta)\cos(\theta)\right)\right] =$$
$$r^3\left[\cos(\theta)\left(2\cos^2(\theta) - 1\right) - 2\sin^2(\theta)\cos(\theta) + i\left(\sin(\theta)\left(1 - 2\cos^2(\theta)\right) + 2\sin(\theta)\cos^2(\theta)\right)\right] =$$
$$r^3\left[\cos(\theta)\left(\left(2\cos^2(\theta) - 1\right) - 2\left(1 - \cos^2(\theta)\right)\right) + i\left(\sin(\theta)\left(\left(1 - 2\sin^2(\theta)\right) + 2\cos^2(\theta)\right)\right)\right] =$$
$$r^3\left[\cos(\theta)\left(4\cos^2(\theta) - 3\right) + i\left(\sin(\theta)\left(\left(1 - 2\sin^2(\theta)\right) + 2\left(1 - \sin^2(\theta)\right)\right)\right)\right] =$$
$$r^3\left[\left(4\cos^3(\theta) - 3\cos(\theta)\right) + i\left(\sin(\theta)\left(3 - 4\sin^2(\theta)\right)\right)\right] =$$
$$r^3\left[\cos(3\theta) + i\sin(3\theta)\right]$$

2.

$$\frac{r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right)}{r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)} = \frac{r_1\left(\cos(\theta_1) + i\sin(\theta_1)\right)}{r_2\left(\cos(\theta_2) + i\sin(\theta_2)\right)} \cdot \frac{\left(\cos(\theta_2) - i\sin(\theta_2)\right)}{\left(\cos(\theta_2) - i\sin(\theta_2)\right)} =$$
$$\frac{r_1}{r_2}\frac{\left(\cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) + i\left(\sin(\theta_1)\cos(\theta_2) - \cos(\theta_1)\sin(\theta_2)\right)\right)}{\left(\cos^2(\theta_2) + \sin^2(\theta_2)\right)} =$$
$$\frac{r_1}{r_2}\left(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right) = \frac{r_1}{r_2}\left(cis(\theta_1 - \theta_2)\right)$$

3. Raise $\left(1 + \sqrt{3}\right)^8$.

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$$\left(1+\sqrt{3}\right)^8 = \left[2\left(\cos\left(\frac{\pi}{3}\right)+i\sin\left(\frac{\pi}{3}\right)\right)\right]^8 = 2^8\left[\cos\left(\frac{8\pi}{3}\right)+i\sin\left(\frac{8\pi}{3}\right)\right]$$
$$= 2^8\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 2^7\left(-1+i\sqrt{3}\right)$$

4. Find $\sqrt[4]{i}$.

$$\sqrt[4]{i} = (0+i)^{1/4} = \left[1\left(\cos\left(\pi/2\right)+i\sin\left(\pi/2\right)\right)\right]^{1/4} = 1\left(\cos\left(\pi/8\right)+i\sin\left(\pi/8\right)\right)$$
$$\left(\frac{\sqrt{2+\sqrt{2}}}{2}\right)+i\left(\frac{\sqrt{2-\sqrt{2}}}{2}\right)$$

5. A certain complex number satisfies $\omega^2 = \omega - 1$. What is ω^{99}? **FURMAN 2001 SR #13.**

$$\omega^2 = \omega - 1 \Rightarrow \omega^2 - \omega + 1 = 0 \Rightarrow \omega = \frac{1 \pm i\sqrt{3}}{2} = \cos\left(\frac{\pi}{3}\right) \pm i\sin\left(\frac{\pi}{3}\right)$$
$$\therefore \omega^{99} = \left(\cos\left(\frac{\pi}{3}\right) \pm i\sin\left(\frac{\pi}{3}\right)\right)^{99} = \cos\left(\frac{99\pi}{3}\right) \pm i\sin\left(\frac{99\pi}{3}\right) = \cos(\pi) \pm i\sin(\pi) = -1$$

6. If $\left(x+iy\right)^3 = -74+ki$, find the absolute value of *k*, given that $x=1$ and $i=\sqrt{-1}$. **NC SMC 2002 INT1.**

$$\left(x+iy\right)^3 = -74+ki \Rightarrow x^3+3x^2(iy)+3x(iy)^2+(iy)^3 =$$
$$x^3-3xy^2+i(3x^2y-y^3) = -74+ki$$

If $x=1$, then $x^3-3xy^2 = 1-3y^2 = -74 \Rightarrow 3y^2 = 75 \Rightarrow y = \pm 5$, and $3\cdot 1^2\left(\pm 5\right)-\left(\pm 5\right)^3 = \pm 15 \mp 125 = -110 \textit{ or } 110$, so the absolute value of *y* is 110.

7. Find the sum of the cube of the roots of the equation $x^{10}+x^9+x^8+\cdots+x^2+x+1=0$ **Duke 2003.**

$x^{10}+x^9+x^8+\cdots+x^2+x+1 = \dfrac{x^{11}-1}{x-1}$, so the roots of the first are exactly the same as the roots of the second, except for x = 1. In the rational expression, we are looking for the eleven 11th roots of unity. If the problem asked for the sum of the roots, this would be easy, since the sum of the roots of $x^{11}-1=0$ is zero, but we can't use 1, so the sum of roots is -1, but that is not the question. We know the roots are evenly spaced around the unit circle at the following angles. Notice that the cubes (hence 3 times the angles) are written underneath.

0	$\frac{2\pi}{11}$	$\frac{4\pi}{11}$	$\frac{6\pi}{11}$	$\frac{8\pi}{11}$	$\frac{10\pi}{11}$	$\frac{12\pi}{11}$	$\frac{14\pi}{11}$	$\frac{16\pi}{11}$	$\frac{18\pi}{11}$	$\frac{20\pi}{11}$
0	$\frac{6\pi}{11}$	$\frac{12\pi}{11}$	$\frac{18\pi}{11}$	$\frac{2\pi}{11}$	$\frac{8\pi}{11}$	$\frac{14\pi}{11}$	$\frac{20\pi}{11}$	$\frac{4\pi}{11}$	$\frac{10\pi}{11}$	$\frac{16\pi}{11}$

We see that the cubes of the complex numbers, as indicated by taking three times the angle, are the same as the original roots, except for the order in which they occur. So the sum of the cubes of the roots is the same as the sum of the roots, which is -1.

8. Find the sum of the squares of the roots of the equation $\frac{x^6-1}{x^2-1}=0$

The roots of the numerator are as follows:

$$\cos 0+i\sin 0,\quad \cos\left(\frac{2\pi}{6}\right)+i\sin\left(\frac{2\pi}{6}\right),\quad \cos\left(\frac{4\pi}{6}\right)+i\sin\left(\frac{4\pi}{6}\right),$$

$$\cos\left(\frac{6\pi}{6}\right)+i\sin\left(\frac{6\pi}{6}\right), \cos\left(\frac{8\pi}{6}\right)+i\sin\left(\frac{8\pi}{6}\right), \cos\left(\frac{10\pi}{6}\right)+i\sin\left(\frac{10\pi}{6}\right)$$

The first zero is simply 1, and the fourth is -1, which we cannot use (because of the denominator), so we are looking at the squares of the other four. These squares are

$$\cos\left(\frac{4\pi}{6}\right)+i\sin\left(\frac{4\pi}{6}\right),\quad \cos\left(\frac{8\pi}{6}\right)+i\sin\left(\frac{8\pi}{6}\right),$$

$$\cos\left(\frac{16\pi}{6}\right)+i\sin\left(\frac{16\pi}{6}\right), \cos\left(\frac{20\pi}{6}\right)+i\sin\left(\frac{20\pi}{6}\right)$$

and these simplify to

$$\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i,\quad \cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i,$$

$$\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{6}\right)=-\frac{1}{2}+\frac{\sqrt{3}}{2}i, \cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$$

, so the sum of these squares is $4\left(-\frac{1}{2}\right)=-2$. If we simply wanted to know the sum of the roots, we could notice that the sum of the roots of the numerator is zero, the sum of the roots of the denominator is also zero, and includes two of the roots of the numerator, so the sum of the roots (not the squares of the roots) of the entire equation is still zero.