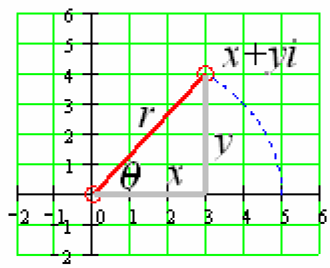
Complex Numbers

Everyone knows that a complex number is a number of the form $x+yi$, where x and y are real numbers and $i=\sqrt{-1}$.



Let's go on from there. It turns out that the polar form of complex numbers is more useful. To convert from the standard or rectangular form $x+yi$ to the polar form, we first find the *norm or modulus* of the number, which is $r=\sqrt{x^2+y^2}$. The *argument or angle* is found using the inverse tangent, $\theta=\begin{cases}\tan^{-1}\left(\frac{y}{x}\right) & x>0\\ \pi+\tan^{-1}\left(\frac{y}{x}\right) & x<0\\ \frac{\pi}{2}, x=0, y>0 \quad \frac{3\pi}{2}, x=0, y<0\end{cases}$.

Once found, the polar form is $r\left(\cos\theta+i\sin\theta\right)$, which is sometimes written $r\cdot cis\theta$. In this form arithmetic of complex numbers becomes less complex.

DeMoivres Theorem really makes things (multiplication, division, powers) simple:

Multiplication:

$$\begin{aligned}&r_1\left(\cos\left(\theta_1\right)+i\sin\left(\theta_1\right)\right)\cdot r_2\left(\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\right)=\\&r_1r_2\left(\cos\left(\theta_1\right)\cos\left(\theta_2\right)+i\sin\left(\theta_1\right)\cos\left(\theta_2\right)+i\sin\left(\theta_2\right)\cos\left(\theta_1\right)-\sin\left(\theta_1\right)\sin\left(\theta_2\right)\right)=\\&r_1r_2\left(\cos\left(\theta_1\right)\cos\left(\theta_2\right)-\sin\left(\theta_1\right)\sin\left(\theta_2\right)+i\left(\sin\left(\theta_1\right)\cos\left(\theta_2\right)+\sin\left(\theta_2\right)\cos\left(\theta_1\right)\right)\right)=\\&r_1r_2\left(\cos\left(\theta_1+\theta_2\right)+i\sin\left(\theta_1+\theta_2\right)\right)\end{aligned}$$

Powers: $\left[r\left(\cos\theta+i\sin\theta\right)\right]^n=r^n\cdot\left(\cos\theta+i\sin\theta\right)^n=r^n\left(\cos\left(n\theta\right)+i\sin\left(n\theta\right)\right)$. The proof of this, for positive integers at least, can be shown with induction and the multiplication theorem.

Roots: The above theorem, when used with fractional exponents, yields the roots of a complex number. $\sqrt[n]{r\left(\cos\left(\theta\right)+i\sin\left(\theta\right)\right)}=r^{1/n}\left(\cos\left(\frac{\theta+2k\pi}{n}\right)+i\sin\left(\frac{\theta+2k\pi}{n}\right)\right), k=0,1,\cdots,n-1$

Example:

$$\sqrt[3]{8}=\sqrt[3]{8\cdot\left(\cos(0)+i\sin(0)\right)}=8^{1/3}\left(\cos\left(\frac{0+2k\pi}{3}\right)+i\sin\left(\frac{0+2k\pi}{3}\right)\right), k=0,1,2$$

$$=2\left(\cos(0)+i\sin(0)\right),2\left(\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right),\left(\cos\left(\frac{4\pi}{3}\right)+i\sin\left(\frac{4\pi}{3}\right)\right)=$$

$$2,\qquad 2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=-1+i\sqrt{3},\qquad 2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=-1-i\sqrt{3}$$

Notice that the three complex cube roots of 8 are all 2 units from the pole, so they are one a circle with radius 2, and they are evenly spaced around that circle at 0, $\frac{2\pi}{3}$, and $\frac{4\pi}{3}$. This will always be the case. The fifth roots of 32, for example, will all have magnitude 2 and arguments 0, $\frac{2\pi}{5},\frac{4\pi}{5},\frac{6\pi}{5}$, and $\frac{8\pi}{5}$.

Division: In multiplication the norms are multiplied and the arguments added. In division the norms are divided and the arguments divided.

$$\frac{r_1 cis\theta_1}{r_2 cis\theta_2}=\frac{r_1}{r_2}cis\left(\theta_1-\theta_2\right)$$

Problems:

1. Show that the power rule works for n = 2, and n = 3.
2. Show that $\frac{r_1 cis\left(\theta_1\right)}{r_2 cis\left(\theta_2\right)}=\frac{r_1}{r_2}cis\left(\theta_1-\theta_2\right)$.
3. Raise $\left(1+\sqrt{3}\right)^8$.
4. Find $\sqrt[4]{i}$.
5. A certain complex number satisfies $\omega^2=\omega-1$. What is ω^{99}? **FURMAN 2001 SR #13.**
6. If $\left(x+iy\right)^3=-74+ki$, find the absolute value of *k*, given that $x=1$ and $i=\sqrt{-1}$. **NC SMC 2002 INT1.**
7. Find the sum of the cube of the roots of the equation $x^{10}+x^9+x^8+\cdots+x^2+x+1=0$ **DUKE 2003**.
8. Find the sum of the squares of the roots of the equation $\frac{x^6-1}{x^2-1}=0$