Different Number Bases

We are so familiar with base-ten numbers that we often forget want a number like 123.456 really means. Of course it means that we have

$$1\cdot 10^2 + 2\cdot 10^1 + 3\cdot 10^0 + 4\cdot 10^{-1} + 5\cdot 10^{-2} + 6\cdot 10^{-3}.$$

In base 10 we must have ten symbols to represent the numbers from 0 to one less than the base (or 9). In a base smaller than 10, say base 5, we only need the digits 0 through 4, and a number like 432.12 means $4\cdot 5^2 + 3\cdot 5^1 + 2\cdot 5^0 + 1\cdot 5^{-1} + 2\cdot 5^{-2}$ which would be $4(25) + 3(5) + 2 + 1(0.2) + 2(0.04) = 117.28_{10}$ as a base ten number.

For bases greater than ten we need additional symbols to stand for all the numbers from 0 to the base minus 1. So in base 16 (hexadecimal) we have the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, A, B, C, D, E, F, and G. The number $F3A4_{16} = 14\cdot 16^3 + 3\cdot 16^2 + 9\cdot 16 + 4 = 58260_{10}$ in base 10.

Of course base 2 (binary) has only the symbols 0 and 1. The base-two number 1101011 would mean (in base ten)

$$1\cdot 2^6 + 1\cdot 2^5 + 0\cdot 2^4 + 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2 + 1 = 64 + 32 + 8 + 2 + 1 = 107_{10}$$

Converting from base ten to another base is fairly easy using the following algorithm. Suppose we want to convert the number 83 to base 2.

$83 = 2(41) + 1$

$41 = 2(20) + 1$

$20 = 2(10) + 0$

$10 = 2(5) + 0$

$5 = 2(2) + 1$

$2 = 2(1) + 0$

$1 = 0(2) + 1$

Now list the remainders in reverse order giving 1010011. A check reveals that $1010011_2 = 2^6 + 2^4 + 2 + 1 = 64 + 16 + 2 + 1 = 83$.

To go from base 2 to base 16, group the digits, from right to left in groups of 4. For this number we have [0101] [0011]. Convert each group of 4 now to base 16, giving $53_{16} = 5\cdot 16^1 + 3\cdot 16^0 = 83_{10}$

Problems:

1. The expression 32 in base b represents the same number as 21 in base c, while the expression 21 in base b represents the same number as 13 in base c. What is b?
2. In what base does $62 \times 14 = 808$?
3. A certain 3-digit number exceeds the sum of its digits by 126. The three-digit number obtained by reversing the order of the digits exceeds the sum of the digits by 225. What is the sum of the digits?
4. The base b numbers 13, 42, and 101 are in arithmetic progression. Find b.
5. A certain fraction r is represented in base b by 0.111111… while in base 2b it takes the simpler form 0.2b. What is r?
6. **How many positive integers less than 1000 have a 1 in their base 16 expansion?** **FURMAN SR 2000 #26**

7. A 6-digit number has its first digit a 9. If you move it to the last digit instead, you get a number which is only one fourth the size of the original number. What is the sum of the digits of the original number? **FURMAN SR 2000 #32**